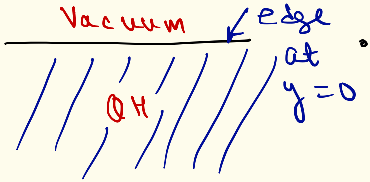
Quantum Hall Edge

Consider a system described by CS theory with a boundary :



Eq. of motion :

Let's first consider e.o.m. in the presence of boundary :

$$\delta S_{CS} = -\frac{k}{4\pi}\int d^3x\ \epsilon_{\mu\nu\lambda}\Big[(\delta a_\mu)(\partial_\nu a_\lambda) + a_\mu(\partial_\nu \delta a_\lambda)\Big]$$

$$= -\frac{k}{4\pi}\int d^3x\ \varepsilon_{\mu\nu\lambda}\Big[\delta a_\mu\, \partial_\nu a_\lambda + \partial_\nu[a_\mu\, \delta a_\lambda] - (\partial_\nu a_\mu)\,\delta a_\lambda\Big]$$

$$= -\frac{k}{4\pi}\int d^3x\ \varepsilon_{\mu\nu\lambda}\Big[\delta a_\mu \underbrace{(\partial_\nu a_\lambda - \partial_\lambda a_\nu)}_{f_{\nu\lambda}} + \partial_\nu[a_\mu\, \delta a_\lambda]\Big]$$

The e.o.m. would be $f_{\nu\lambda}=0$ if the total derivative term, that equals,

$-\frac{k}{4\pi}\int dx\, dt\ [a_x\, \delta a_t - a_t\, \delta a_x]$ vanishes.

One can do this by either setting $a_t = 0$ or $a_x = 0$ at the boundary $y=0$. More generally, one can chose a linear combination $a_t - v a_x \big|_{y=0} = 0$. The parameter v will turn out to be the velocity of chiral edge modes.

Guage invariance:

The action is <u>not</u> guage invariant now if we allow arbitrary guage transformations: $a \to a + d\omega$

$$\Rightarrow \quad S_{cs} = -\frac{k}{4\pi}\int a\, da \to -\frac{k}{4\pi}\int (a + d\omega)\, da$$

$$= -\frac{k}{4\pi}\int a\, da - \frac{k}{4\pi}\int_{y=0} dx\, dt \;\; \omega\, [\partial_t a_x - \partial_x a_t]$$

$$= S_{cs} - \frac{k}{4\pi}\int_{y=0} dx\, dt \;\; \omega\, [\partial_t a_x - \partial_x a_t]$$

For S_{cs} to be guage invariant, one may require that $\omega = 0$ at the boundary $(y=0)$. Restricting guage transformations leads to new physical d.o.f.

To derive an action for these physical d.o.f that live at the boundary, lets extend the boundary condition $a_t - v a_x = 0$ to the bulk.

It's easier to implement this via a change of coordinates: $t' = t$, $x' = x + vt$, $y' = y$.

so that $a'_{t'} = a_t - v a_x$, $a'_{x'} = a_x$, $a'_{y'} = a_y$

$$\Rightarrow \quad \boxed{a'_{t'} = 0}$$

Thus $a'_{t'}$ can't enter the action and only acts as a lagrange multiplier in the action, thus imposing $f'_{x'y'} = 0$

$$\Rightarrow \quad a'_{x'} = \partial_{x'} \varphi, \quad a'_{y'} = \partial_{y'} \varphi.$$

The action becomes

$$S_{CS} = -\frac{k}{4\pi} \int d^3 r' \, \varepsilon^{ij} a'_i \partial_{t'} a'_j$$

where $i, j = x', y'$.

$$= -\frac{k}{4\pi} \int d^3r' \; \partial_{x'}\varphi \, \partial_{t'} \partial_{y'}\varphi - \partial_{y'}\varphi \, \partial_{t'}\partial_{x'}\varphi$$

$$= -\frac{k}{4\pi} \int dx' \, dt' \left[\partial_{t'}\varphi \; \partial_{x'}\varphi \right] \Big|_{y=0}$$

$$= -\frac{k}{4\pi} \int dx \, dt \; \left[\partial_t \varphi \, \partial_x \varphi - v (\partial_x \varphi)^2 \right]$$

This look a bit strange, but precisely corresponds to a chiral edge mode:

$$\text{e.o.m.} \Rightarrow \quad \partial_t \partial_x \varphi - v \, \partial_x^2 \varphi = 0$$

Defining $\rho = \dfrac{\partial_x \varphi}{2\pi} \Rightarrow$

$$\boxed{\partial_t \rho - v \, \partial_x \rho = 0}$$

which is a traveling wave along $+x$ direction, with speed v.

$\rho =$ electron density

Coupling the system to E.M. field adds a term $\Delta S = \int d^3x \; A_\mu J^\mu = \frac{1}{2\pi}\int d^3x \; A_\mu \, \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$ to the action. Integrating by parts, and assuming that only A_t, A_x are non-zero and are only a f^n of (x,t), one finds:

$$\Delta S = \frac{1}{2\pi} \int dx\, dt \, [A_t - v A_x] \partial_x \phi$$

$$\Rightarrow \quad \frac{\partial_x \phi}{2\pi} = e^- \text{ density} \equiv \rho$$

and $\quad v \dfrac{\partial_x \phi}{2\pi} =$ current along x.

Recall the continuity eqn. is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

$$\Rightarrow \quad \partial_x \partial_t \phi + v \partial^2_x \phi = 0$$

which is indeed the e.o.m. discussed above.

Compactness of φ : $\varphi \equiv \varphi + 2\pi$

Consider the total charge on the boundary, assuming that the boundary has a circumference L :

$$Q = \int_0^L d\sigma\, \rho = \frac{1}{2\pi} \int d\sigma \frac{\partial \varphi}{\partial \sigma} .$$

For this to be an integer, one requires that φ winds around by $2\pi Q$. Thus φ behaves like a phase. Another way to see this is that $a_i = \partial_i \varphi$ and a is a compact guage field (the physical variables are $e^{i\oint a \cdot dl}$). This again implies that φ is compact. We will call it a 'chiral boson' for obvious reasons.

Quantizing Chiral Boson:

$$S = \frac{-k}{4\pi}\int\left[\partial_x \varphi\ \partial_t\varphi - v(\partial_x\varphi)^2\right]$$

Fourier transform:

$$\varphi(\sigma,t) = \frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty}\varphi_n(t)\, e^{2\pi i n\sigma/L}$$

$$\left(k_n = \frac{2\pi n}{L}\right)$$

$$\Rightarrow\quad S = -\frac{k}{4\pi}\int dt \sum_{n=-\infty}^{\infty}\Big[i k_{-n}\,\dot{\varphi}_n\,\varphi_{-n}$$

$$+\ v\, k_n\, k_{-n}\,\varphi_n\,\varphi_{-n}\Big]$$

with $\quad \varphi^*_n = \varphi_{-n}$

$$= \frac{k}{2\pi}\int dt \sum_{n=0}^{\infty}\Big[i k_n\,\dot{\varphi}_n\,\varphi_{-n} + v\,k^2_n\,\varphi_n\,\varphi_{-n}\Big]$$

The zero mode decouples from the dynamics, and we will set it equal to zero.

$$\Rightarrow\quad [\varphi_n, \varphi_{n'}] = \frac{2\pi}{k}\,\frac{1}{k_n}\,\delta_{n+n'}$$

$$[\rho_n, \varphi_{n'}] = \frac{i}{k} \delta_{n+n'}$$

$$[\rho_n, \rho_{n'}] = \frac{kn}{2\pi k} \delta_{n+n'}$$

In real-space:

$$[\varphi(\sigma), \varphi(\sigma')] = \frac{i\pi}{m} \operatorname{sign}(\sigma-\sigma')$$

$$[\rho(\sigma), \varphi(\sigma')] = \frac{i}{m} \delta(\sigma-\sigma')$$

$$[\rho(\sigma), \rho(\sigma')] = -\frac{i}{2\pi k} \partial_\sigma \delta(\sigma-\sigma')$$

Hamiltonian $H = \dfrac{kv}{2\pi} \displaystyle\sum_{n=0}^{\infty} k^2 n \, \varphi_n \varphi_{-n}$

$$= 2\pi k v \sum_{n=0}^{\infty} \rho_n \rho_{-n}$$

Where is electron (again)?

Consider the operator $\psi = :e^{ik\varphi}:$ where $:\ :$ denotes normal ordering.

Using commutation rel. for φ

$$\psi(\sigma)\,\psi(\sigma') = e^{-k^2[\varphi(\sigma),\varphi(\sigma')]}\,\psi(\sigma')\,\psi(\sigma)$$

$$= e^{-k^2 \frac{i\pi}{k}\,\mathrm{sign}(\sigma-\sigma')}\,\psi(\sigma')\,\psi(\sigma)$$

When m is odd $e^{-i\pi k\,\mathrm{sign}(\sigma-\sigma')} = -1$

$$\Rightarrow \quad \{\psi(\sigma),\psi(\sigma')\} = 0$$

Similarly, when k is even $[\psi(\sigma),\psi(\sigma')] = 0$

One can also check:

$$[\rho(\sigma),\psi^+(\sigma')] = \psi^+(\sigma')\,\delta(\sigma-\sigma')$$

Since $\rho \sim \psi^+\psi$

$$\psi^+\psi\psi^+ - \psi^+\psi\psi^+ = \psi^+$$

Thus, commutation relations work out.

Electron's Green fn :

$$G_F(x+vt) = \langle \psi^+(x,t)\, \psi(0,0) \rangle$$

$$= e^{k^2 \langle \phi(x,t)\, \varphi(0,0) \rangle}$$

But $\langle \varphi(x,t)\, \varphi(0,0) \rangle = -\frac{1}{k} \log(x+vt)$

$$\Rightarrow \quad G_F(x+vt) \sim \frac{1}{(x+vt)^k}$$

Parton Construction of FQH states

The Laughlin wave-f^n at $\nu = 1/k$ $\prod_{i<j}(z_i - z_j)^k e^{-|z|^2}$,

is essentially k'th power of wave-f^n for integer QH (i.e. $\nu=1$).

Similarly, the edge state correlation f^n in the $\nu = 1/k$ state, $\langle \psi^+(x,t)\psi(0,0)\rangle \sim \frac{1}{(x+vt)^k}$, is the k'th power of the correlation f^n at the edge state for $\nu = 1$.

These observations motivate a parton construction of the form

$$c = f_1 f_2 \ldots f_k$$

where each f_i is in a integer q.h. state at $\nu = 1$.

Let's demonstrate this for bosonic f.q.h. at $\nu = 1/2$.

writing $b = f_1 f_2$, there is a guage redundancy $f_1 \to f_1 e^{i\theta}$, $f_2 \to f_2 e^{-i\theta}$ $\Rightarrow$ f_1, f_2 are coupled to an internal guage field a with opposite charges. Further, the electromagnetic charge of f_1 and f_2 should add upto that of b, which we take to be one. let's assign f_1 an E.M. guage charge of q, and f_2 $1-q$.

The parton Hamiltonian is

$$H = \sum_\alpha f^+_\alpha \, -i\partial_t f_\alpha$$

$$+ \sum_\alpha \frac{1}{2m} f^+_\alpha \left[-i\partial_x - q_\alpha \vec{a} - Q_\alpha \vec{A} \right]^2 f_\alpha$$

$\alpha = 1, 2$, $q_1 = 1$, $q_2 = -1$, $Q_1 = q$, $Q_2 = 1-q$.

Since both f_1 and f_2 are in IQH at $\nu = 1$, integrating out f_1, f_2 yields:

$$\mathcal{L} = \frac{1}{4\pi} [-a + (1-q) A] \, d \, [-a + (1-q) A]$$

$$+ \frac{1}{4\pi} [a + qA] \, d \, [a + qA]$$

$$= \frac{2 \, a da}{4\pi} + \frac{AdA}{4\pi} [q^2 + (1-q)^2]$$

$$+ \frac{adA}{4\pi} [-2(1-q) + 2q]$$

$$= \frac{2ada}{4\pi} + \frac{AdA}{4\pi} [q^2 + (1-q)^2] + \frac{adA}{4\pi} [2 - 4q]$$

To find Hall response, consider the e.o.m. for a:

$$4 \, da + dA [2 - 4q] = 0$$

$$\Rightarrow \quad da = -\frac{dA}{2} (1 - 2q)$$

Substituting this back,

$\mathcal{L} =$

$$\frac{A\,dA}{4\pi}\left[\frac{2}{4}(1-2q)^2 + \left[q^2+(1-q)^2\right] - \frac{1}{\cancel{2}}(1-2q)\,\cancel{2}\,(1-2q)\right]$$

$$= \frac{1}{2}\,\frac{A\,dA}{4\pi}$$

$$\Rightarrow \quad J = \frac{\delta \mathcal{L}}{\delta A} = \frac{1}{2}\,\frac{dA}{2\pi}$$

$$\Rightarrow \quad \sigma_{xy} = \frac{1}{2}.$$

Further, g.s.d. on $T^2 = 2$ since the low-energy theory in the absence of A is $\dfrac{2\,ada}{4\pi}$.

The statistics of quasiparticle with unit charge is $\dfrac{\pi Q^2}{k} = \dfrac{\pi}{2} \times 1^2 = \dfrac{\pi}{2}$.

This is precisely the (single) semion theory.